Breakdown of the KLN Theorem for Charged Particles in Condensed Matter

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Abstract. The Kinoshita-Lee-Nauenberg (KLN) theorem describes the fact that inclusive electromagnetic and weak production processes in the vacuum do not contain singularities in the ultra-relativistic limit of zero mass. When these production processes occur in condensed matter, the KLN theorem fails. One consequence of this failure is that precision lifetime determinations of stopped muons will depend on the nature of the surrounding material.

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1. Introduction

The KLN theorem (Kinoshita 1962, Lee and Nauenberg 1964) refers to the removable nature of singularities in the production probabilities of particles in the limit of zero mass. For example, that the photon has zero mass in the vacuum led to the quantum electrodynamic "infrared catastrophe" wherein more and more photons were created at lower and lower frequencies. The catastrophe was cured with the realization that photon detectors have finite energy resolution. If the resolution was kept finite as a formal photon mass went to zero, no infinities appeared in the final results.

A generalization may be made for ultra-relativistic charged particles wherein the mass appears as a small parameter. For example, consider the weak decay

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu. \tag{1}$$

The vacuum *radiatively corrected* width (to lowest orders in G_F and α) is given by (Kinoshita T and Sirlin A 1959, Berman S 1958, Lee T D 1988)

$$\Gamma_\mu = \Gamma_\mu^{(0)} \left\{ 1 - \frac{\alpha}{2\pi}\left(\pi^2 - \frac{25}{4}\right) - \left(\frac{8m^2}{M^2}\right) + \ ... \right\}, \tag{2}$$

where $\Gamma_\mu^{(0)} = (Mc^2/192\pi^3\hbar)(G_F M^2/\hbar c)^2$, M is the muon mass, m is electron mass, and $\alpha = (e^2/\hbar c)$. Note, in the fictitious zero electron mass limit $m \to 0$, there is no singularity in Γ_μ. In its general form, the KLN theorem protects the production probability (in the vacuum) from zero mass limit singularities.

However, when a muon decays from rest in condensed matter (as in laboratory measurements of Γ_μ) the vacuum KLN protection no longer holds true. This becomes evident from the retardation force F in the condensed matter felt by an ultra-relativistic electron. It is well known theoretically (Berestetskii V B *et. al.* 1994) and experimentally (Caso C et. al. 1998) that

$$F \approx \left(\frac{e^2\omega_p^2}{c^2}\right)\left\{ ln\left(\frac{2E^2}{Imc^2}\right) - 1 + \ ...\right\}, \quad E >> mc^2, \tag{3}$$

where I is the log mean *ionization potential* of the atoms, and the plasma frequency

$$\omega_p^2 = \left(\frac{4\pi n e^2}{m}\right). \tag{4}$$

The finite number density n of electrons in the material is responsible for decelerating the fast electron in accordance with Eq.(3), and by a similar process is responsible for stopping the muon in the first place. Clearly, the occurrence of the electron mass in the logarithm of Eq.(3) precludes a smooth limit $m \to 0$ in the retardation force F for ultra-relativistic particles. The KLN violation is present as long as the density of electrons in the condensed matter is finite. This is the first instance for which the KLN theorem for decays in condensed matter fails.

In more detail, the wave function $\psi(x)$ for a fast lepton of mass $(\hbar\kappa/c)$ traveling through a condensed matter medium obeys a non-local Dirac equation of the form (Dyson F J 1949, Schwinger J 1951)

$$(-i\gamma^\mu\partial_\mu + \kappa)\,\psi(x) + \int \Sigma(x-y)\psi(y)d^4y = 0, \tag{5}$$

where the "self energy" $\Sigma(x-y)$ depends on the continuous medium and is certainly *not* well approximated by a local potential. The fact that condensed matter induces a non-locality in the self energy part is merely another description of the ability of

condensed matter to stop a high energy particle. To lowest order in α, the ultra-relativistic lepton has a self energy

$$\Sigma(x-y) = i\alpha\gamma^{\mu}S(x-y)\gamma^{\nu}\mathcal{D}_{\mu\nu}(x-y) + \ ..., \tag{6}$$

where $S(x-y)$ obeys the free lepton Dirac equation

$$\left(-i\gamma^{\mu}\partial_{\mu} + \kappa\right) S(x-y) = \delta(x-y), \tag{7}$$

and the photon propagator *in the condensed matter medium* has the form

$$\mathcal{D}_{\mu\nu}(x) = \int D_{\mu\nu}(Q)e^{iQ\cdot x}\frac{d^4Q}{(2\pi)^4}\ . \tag{8}$$

In Sec.2, we shall discuss the form of the photon propagator $D_{\mu\nu}(Q)$ in continuous media described by a dielectric response function $\varepsilon(\zeta)$ for complex frequency ζ. In Sec.3, it will be shown how the retardation force on an ultra-relativistic lepton can be related to the condensed matter induced non-local self energy part $\Sigma(x-y)$. This induced self energy renders the KLN protection in the limit $\kappa \to 0$ invalid. In the concluding Sec.4, we explore a notion of how KLN violating condensed matter effects can be made manifest in transition rates, e.g. Γ_{μ}.

2. Electromagnetic Fields in Continuous Media

The photon propagator in a material medium will be reviewed. All discussions of electromagnetic fields in materials should start with the Maxwell's equations

$$\partial_{\mu}\ {}^{*}F^{\mu\nu} = 0, \tag{9}$$

$$\partial_{\mu}F^{\mu\nu} = -\left(\frac{4\pi}{c}\right)J^{\nu}. \tag{10}$$

The current may be broken up into a polarization current plus an external current

$$J^{\nu} = c\partial_{\mu}P^{\mu\nu} + J^{\nu}_{ext}. \tag{11}$$

Eq.(11) reads as $\rho = -div\mathbf{P} + \rho_{ext}$ and $\mathbf{J} = c\ \mathbf{curl}\ \mathbf{M} + (\partial\mathbf{P}/\partial t) + \mathbf{J}_{ext}$, in a specific Lorentz frame. If we now define

$$H^{\mu\nu} = F^{\mu\nu} + 4\pi P^{\mu\nu}, \tag{12}$$

then Eqs.(10)-(12) imply

$$\partial_{\mu}H^{\mu\nu} = -\left(\frac{4\pi}{c}\right)J^{\nu}_{ext}. \tag{13}$$

Just as $F^{\mu\nu}$ yields $\mathbf{E}$ and $\mathbf{B}$, we have $H^{\mu\nu}$ yielding $\mathbf{D} = \mathbf{E}+4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B}-4\pi\mathbf{M}$ via Eq.(12). Maxwell's Eqs.(9) and (13) determine the electromagnetic fields in material media. However the electromagnetic properties of the medium must be included as constitutive equations (in linear order) to compute the photon propagator.

The constitutive equations are most easily expressed in the rest frame of the condensed matter. However, we here take the option of employing a strictly covariant notation. This may be done by considering the stress tensor $T_{\mu\nu}$ for the material. The stress tensor has four eigenvalues found from $det||T^{\nu}_{\mu} - P\delta^{\nu}_{\mu}|| = 0$. Three of the eigenvalues (P_I, P_{II}, P_{III}) correspond to possible values of the material pressure in three spatial eigenvector directions. The fourth eigenvalue is determined by the material mass density $(P_{IV}/c^2) = -\rho_{mass}$ and is described by a time-like eigenvector

$$N_{\mu}N^{\mu} = -1, \tag{14}$$

$$T_\mu^\nu N_\nu = -(\rho_{mass} c^2) N_\mu. \tag{15}$$

The dielectric response is a non-local (in space-time) linear operator $\hat{\varepsilon}$ describing

$$(N_\mu H^{\mu\nu}) = \hat{\varepsilon}(N_\mu F^{\mu\nu}), \tag{16}$$

i.e. $\mathbf{D} = \hat{\varepsilon}\mathbf{E}$ in the rest frame of the material. We shall further assume that

$$N_\mu \, {}^*H^{\mu\nu} = N_\mu \, {}^*F^{\mu\nu} \tag{17}$$

i.e. $\mathbf{H} = \mathbf{B}$ in the rest frame of the material.

Thus far our description is *strictly gauge invariant* since we have not yet introduced vector potentials, and manifestly Lorentz covariant. However, in order to discuss the photon propagator one must choose a vector potential in a particular gauge. One satisfies Eq.(9) by introducing a vector potential A^μ via

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{18}$$

In the vacuum, it is often convenient to employ the covariant (Lorentz) gauge $\partial_\mu A_L^\mu = 0$. The close analogue to the covariant gauge is the material Lorentz gauge. In the condensed matter rest frame, with $N^\mu = (\mathbf{0}, 1)$ and $A^\mu = (\mathbf{A}, \phi)$, we fix the gauge according to

$$\frac{1}{c}\left(\frac{\partial(\hat{\varepsilon}\phi)}{\partial t}\right) + div\mathbf{A} = 0. \tag{19}$$

In this gauge Eqs.(13) and (16)-(18) read

$$\left\{\frac{\hat{\varepsilon}}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2\right\}\phi = \left(\frac{4\pi}{\hat{\varepsilon}}\right)\rho_{ext}, \tag{20}$$

and

$$\left\{\frac{\hat{\varepsilon}}{c^2}\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2\right\}\mathbf{A} = \left(\frac{4\pi}{c}\right)\mathbf{J}_{ext}. \tag{21}$$

From Eqs.(20) and (21), we find the material Lorentz gauge photon propagator in the space like directions

$$D_{ij}(\mathbf{Q}, \omega) = \left(\frac{4\pi\delta_{ij}}{\big(|\mathbf{Q}|^2 - \varepsilon(|\omega| + i0^+)(\omega/c)^2 - i0^+\big)}\right), \tag{22}$$

and in the time-like direction

$$D_{00}(\mathbf{Q}, \omega) = \left(\frac{4\pi}{\varepsilon(|\omega| + i0^+)\big(|\mathbf{Q}|^2 - \varepsilon(|\omega| + i0^+)(\omega/c)^2 - i0^+\big)}\right). \tag{23}$$

The expressions in Eqs.(22) and (23) for the photon propagator in matter in the gauge of Eq.(19) are well known (Abrikosov A A *et. al.* 1963). These have been fully discussed for materials, along with the other standard gauges (e.g. radiation or temporal) with regard to the statistical physics of radiation in matter (Lifshitz E M and Pitaevskii L P 1984).

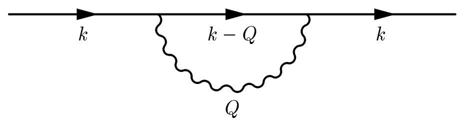

Figure 1. The self energy of a lepton is shown to lowest order in α.

3. The Lepton Self Energy

In k-space, the self energy of Eqs.(6) and (7) has the form shown in Fig.1; It is

$$\Sigma(k) = i\alpha \int \gamma^\mu \left(\frac{1}{\kappa + \gamma\cdot(k-Q) - i0^+} \right) \gamma^\nu D_{\mu\nu}(Q) \frac{d^4Q}{(2\pi)^4} \ . \tag{24}$$

If an ultra-relativistic charged lepton having four velocity v is moving through the condensed matter and is described by a Dirac spinor u, then the transition rate per unit proper time Γ_γ to emit a photon in the medium is determined by the imaginary part of the self energy; i.e. $\Gamma_\gamma = -2c\Im m\big(\bar{u}\Sigma(k = \kappa v/c)u\big)$. This amounts to the quasi-classical Dirac matrix replacement $\gamma^\mu \to (v^\mu/c)$. Thus Eq.(24) yields a photon production rate

$$\Gamma_\gamma = 2\pi\alpha \int v^\mu \delta(v\cdot Q) v^\nu \Im m\left(D_{\mu\nu}(Q)\right) \frac{d^4Q}{(2\pi)^4} \tag{25}$$

per unit proper time of the ultra-relativistic particle.

It is of interest to compare our derivation of the photon production rate in Eq.(25), via $\Sigma(k)$, to an earlier derivation (Schwinger J 1976) which starts from the non-local action for the current,

$$S = \left(\frac{1}{2c^3}\right) \int\int J^\mu(x) J^\nu(y) \mathcal{D}_{\mu\nu}(x-y) d^4x d^4y. \tag{26}$$

If one starts from a "classical current" representing the ultra-relativistic lepton

$$J^\mu(x) = ec \int \delta(x - v\tau) v^\mu d\tau, \tag{27}$$

then radiated photons are created with a Poisson distribution $p_n = (n^{N_\gamma}/n!)e^{-N_\gamma}$. The mean number of radiated photons is N_γ. The probability of radiating zero photons $p_0 = |e^{iS/\hbar}|^2 = e^{-N_\gamma}$ yields $N_\gamma = \Im m(2S/\hbar)$; i.e.

$$N_\gamma = \Im m \left(\frac{1}{\hbar c^3} \int\int J^\mu(x) J^\nu(y) \mathcal{D}_{\mu\nu}(x-y) d^4x d^4y \right). \tag{28}$$

With $L^\mu(Q,\tau) = \int_{(-\tau/2)}^{(\tau/2)} \left(v^\mu e^{iQ\cdot v\tau'}\right) d\tau'$, Eqs.(8), (27) and (28) yield

$$N_\gamma = \alpha \Im m \left(\int L^\mu(Q,\tau) L^\nu(Q,\tau)^* D_{\mu\nu}(Q) \frac{d^4Q}{(2\pi)^4} \right). \tag{29}$$

Eq.(29) yields the same number of radiated photons per unit proper time

$$\Gamma_\mu = \lim_{\tau\to\infty} \left(\frac{N_\gamma}{\tau}\right) \tag{30}$$

as does Eq.(25), which was based on the self energy part $\Sigma(k)$ in Eq.(24).

In the rest frame of the material, employing a laboratory time T,

$$\Gamma_\mu^{lab} = \lim_{T\to\infty}\left(\frac{N_\gamma}{T}\right) = \Gamma_\mu\sqrt{1-\left(\frac{|\mathbf{v}|}{c}\right)^2}\ \ , \tag{31}$$

it follows from Eqs.(22), (23), (25) and (30) that

$$\left(\frac{d\Gamma_\mu^{lab}}{d\omega}\right) = 4\pi\alpha\mathcal{K}(\omega), \quad \text{restricted to } (\omega > 0), \tag{32}$$

$$\mathcal{K}(\omega) = 2\Im m\int\left\{\frac{\delta(\omega-\mathbf{Q}\cdot\mathbf{v})\big((\mathbf{v}/c)^2-\varepsilon^{-1}(\omega+i0^+)\big)}{\big(|\mathbf{Q}|^2-\varepsilon(\omega+i0^+)(\omega/c)^2-i0^+\big)}\right\}\frac{d^3\mathbf{Q}}{(2\pi)^3}\ . \tag{33}$$

To calculate the retardation force $\mathbf{F}$ one may compute the power loss according to

$$\mathbf{F}\cdot\mathbf{v} = -\int_0^\infty \hbar\omega\left(\frac{d\Gamma_\mu^{lab}}{d\omega}\right)d\omega = \tag{34}$$

$$-\left(\frac{2e^2}{\pi v c^2}\right)\Im m\int_0^\infty\int_0^{q_{max}}\left\{\frac{\big((v/c)^2-\varepsilon^{-1}\big)\omega q dq d\omega}{\big(q^2-(\varepsilon-(c/v)^2)(\omega/c)^2-i0^+\big)}\right\}. \tag{35}$$

In Eqs.(34) and (35) we have decomposed $d^3\mathbf{Q} = d^2\mathbf{q}dQ_{||}$, where $Q_{||} = (\mathbf{Q}\cdot\mathbf{v}/|\mathbf{v}|)$, and introduced a high wave number cutoff q_{max}.

After playing many ingenious games with logarithms, Eq.(35) in combination with the plasma frequency sum rule

$$\left(\frac{2}{\pi}\right)\int_0^\infty \omega\Im m\varepsilon(\omega+i0^+)d\omega = \omega_p^2, \tag{36}$$

yields a high energy retardation force formula in agreement with the standard result of Eq.(3),

$$F \to \left(\frac{e^2\omega_p^2}{c^2}\right)\left\{ln\left(\frac{2E^2}{Imc^2}\right) - 1 + \ ...\right\} \ \text{ as } \ (E/mc^2)\to\infty. \tag{37}$$

The fact that the logarithm in unavoidable in retardation force calculations in both theory and laboratory experiments clearly shows the failure of the KLN theorem in condensed matter. The theorem is (of course) valid and useful for particles moving through the vacuum (Sterman G and Weinberg S 1977). But the theorem cannot properly be applied to experiments wherein particles move through materials.

4. Conclusions

We have shown that the standard condensed matter retardation force formulas follow from the self energy part $\Sigma(k)$ of a charged lepton. The lowest order in α correction is shown in Fig.1. Furthermore, the structure of the inclusive transition rate to excite the condensed matter, i.e. $\big(-2c\Im m\Sigma(k)\big)$, violates the required hypotheses of the KLN theorem. This violation has profound implications for ultra-high precision measurements of lepton lifetimes. For example, consider the decays both with and without real photon radiation. If the photons are merely virtual, as in Figs.3, 4 and 5, then the reaction is in reality

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e. \tag{38}$$

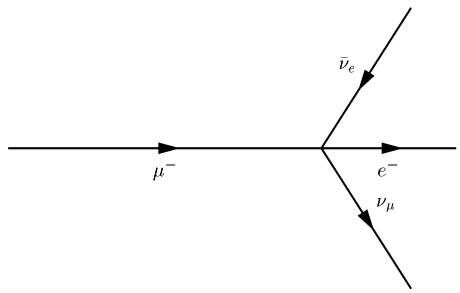

Figure 2. The decay of a muon is shown in the bare Fermi theory.

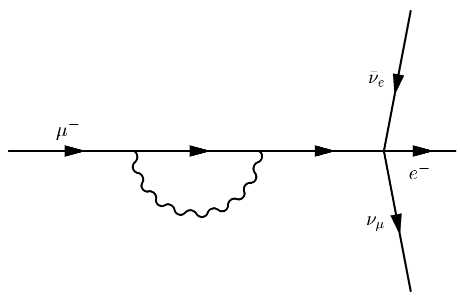

Figure 3. The self energy of the muon to lowest order in α.

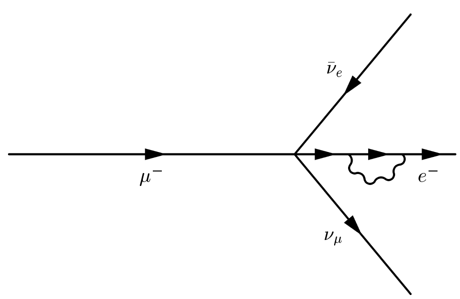

Figure 4. The self energy of the electron to lowest order in α.

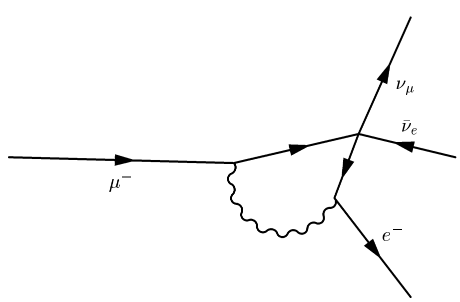

Figure 5. The muon-electron vertex correction to lowest order in α.

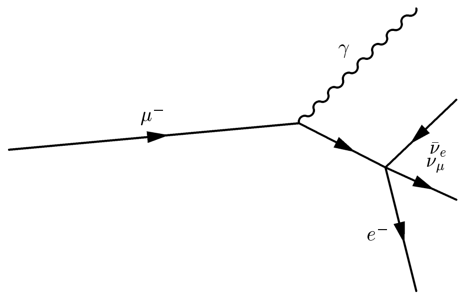

Figure 6. The muon emits a real photon.

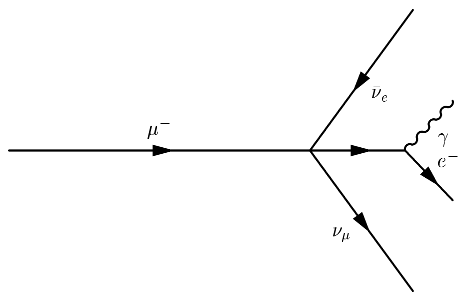

Figure 7. The electron emits a real photon.

From the interference of the superposition of amplitudes shown in Figs.3, 4 and 5, with the amplitude shown in Fig.2, one finds a *decrease* in the total muon decay rate. On the other hand there is a somewhat smaller increase in the total muon decay rate due the emission of real photons as shown in Figs.6 and 7. The real photon emission reaction is

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e + \gamma. \tag{39}$$

Both Eqs.(38) and (39) should include radiative corrections of order α. The total effect to order α is to *decrease* the total decay rate of the muon.

In order to properly include initial and final state interactions, the self energy parts $\Sigma(k)$ of both the muon and the electron, i.e. the diagrams both in Figs.3 and 4 must be calculated *including condensed matter effects* as fully discussed in this work. It makes *no sense* to ignore the simple experimental fact that charged particles may be stopped in condensed matter via the induced lepton self energy parts $\Sigma(k)$.

Once this simple experimental fact is realized, it becomes evident (for reasons of gauge invariance) that all photons, virtual or real, must be renormalized by the condensed matter. Since all photons propagate as Eqs.(20) and (21), condensed matter effects must be included in the final state interactions of the photon. Thus, the condensed matter corrections up to order α shown become somewhat larger than the vacuum terms of order α^2.

The breakdown of the KLN theorem in condensed matter dictates that internal electromagnetic fields must be seriously considered in precision computations of experimental muon lifetimes. The condensed matter electric fields are sufficiently large to actually stop an energetic lepton. This would be most difficult to do in a vacuum.

The requirement of considering condensed matter effects in precision determinations of the muon lifetime is most easily proved (or disproved) by experimental studies in which the muons are stopped in different sorts of materials.

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